PERGAMON

International Journal of Solids and Structures 36 (1999) 4425–4445

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Order of singularity and singular stress field about an axisymmetric interface corner in three-dimensional isotropic elasticity

Yi-Hua Liu[a], Jin-Quan Xu[b,*], Hao-Jiang Ding[c]

[a] *Department of Applied Mathematics and Mechanics, Hefei University of Technology, Hefei, 230009, P.R. China*
[b] *Department of Mechanics, Zhejiang University, Hangzhou, 310027, P.R. China*
[c] *Department of Civil Engineering, Zhejiang University, Hangzhou, 310027, P.R. China*

Received 28 October 1997; in revised form 22 June 1998

Abstract

The spatial axisymmetric problem of an isotropic, elastic, homogeneous body containing an inclusion of a different material with an interface corner point (along a circular contour) and arbitrary joining angles is considered in this paper. It is found that the order of the stress singularity at the interface corner coincides with that of the corresponding plane strain problem, but the dependence of the singular stress field on the material properties depends on both the Poisson ratios (of the two isotropic materials) as well as on the ratio of their shear moduli. The theoretical results have been confirmed by numerical, finite-element-based results in a special bimaterial case. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Axisymmetric; Bimaterial; Contact; Corner; Displacement; Elasticity; Inclusions; Interface; Isotropic; Singularities; Stress

1. Introduction

The order of the stress singularity at an interface corner of bonded dissimilar materials in plane elasticity problems was first studied by Bogy and Wang (1971). More recently, Chen and Nisitani (1993) studied the same problem again and they derived both the eigenequations for the orders of the stress singularity as well as the singular stress field near the interface corner by separating the whole problem into the symmetric and the skew-symmetric cases. On the other hand, for three-dimensional axisymmetric interface corner problems, only some special cases have been studied so far. Bažant and Keer (1974) examined the axisymmetric vertex of conical notches and rigid inclusions. Keer and Parihar (1978) studied the axisymmetric vertex of a conical inclusion bonded to a conical notch. Atkinson et al. (1982) investigated a rod pull out problem, whereas Muki and

* Corresponding author. Fax: 0086 571 7951358; e-mail: mecxujq@public.zju.edu.cn

PII: S0020-7683(98)00200-5

Sternberg (1970) and Pak and Gobert (1993) studied the problem of a partially embedded rod. Kohno and Ishikawa (1991) and Chen and Nisitani (1992a, b) dealt with the fiber-end problem in short fiber-reinforced composites by using an approximation referring to the plane problem. Furthermore, Picu (1996) analyzed the stress singularity at a vertex of a conical inclusion with a freely sliding interface. Somaratna and Ting (1986a, b) examined the stress singularity at the apex of a conical notch and of a conical rigid inclusion in transversely isotropic materials. Fleck and Durban (1989) gave the asymptotic field at the tip of a conical notch and of a rigid cone in an incompressible power law hardening solid. Some additional related investigations are also available in the literature.

There are several applicable approaches for the singularity problems in the axisymmetric elasticity so far. They may fall into the following category:

(1) Series expansion techniques, including: (a) series expansions of separation of the variables in terms of power series (Zak, 1972; Naolake and Tomoaki, 1992); (b) the eigenfunction expansions by using biorthogonality or optimal weighting functions (Robert and Keer, 1987); (c) the series expansion in terms of Dini series (Kim and Steele, 1992), a method extended from the singularity problems in the plane elasticity recently (Kim and Steele, 1990).
(2) Reduction to singular integral equations in terms of integral transform techniques (Muki and Sternberg, 1970; Gupta, 1974; Atkinson et al., 1982; Gecit, 1986; Pak and Gobert, 1993).
(3) Separation of variables in terms of the Legendre functions for the conical notch and inclusion problems (Bažant and Keer, 1974; Keer and Parihar, 1978; Somaratna and Ting, 1986a).
(4) Finite element methods (Somaratna and Ting, 1986b).

Because many additional axisymmetric interface corner problems are also of interest in applied mechanics and engineering, this paper investigates the general problem of the spatial axisymmetric interface corner with arbitrary joining angles, as shown in Fig. 1, by using the series expansion technique of separation of the variables in terms of power series in order to obtain not only the order of the stress singularity but also the displacement and singular stress fields. Here $\theta_1+\theta_2 = 2\pi$, z is the axisymmetric axis, ρ is the radial direction and a ($a \neq 0$) is the distance between the corner O' and the axisymmetric axis z. This problem may have a wide applicability in engineering practice

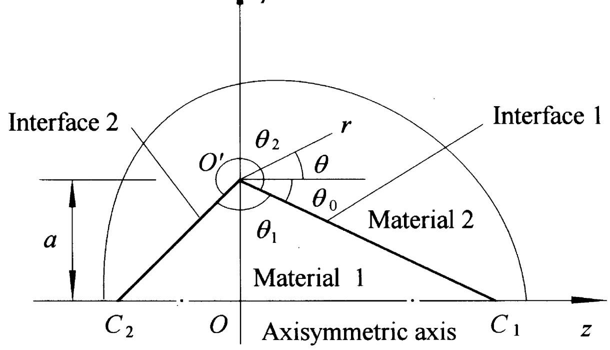

Fig. 1. General model of the axisymmetric interface corner.

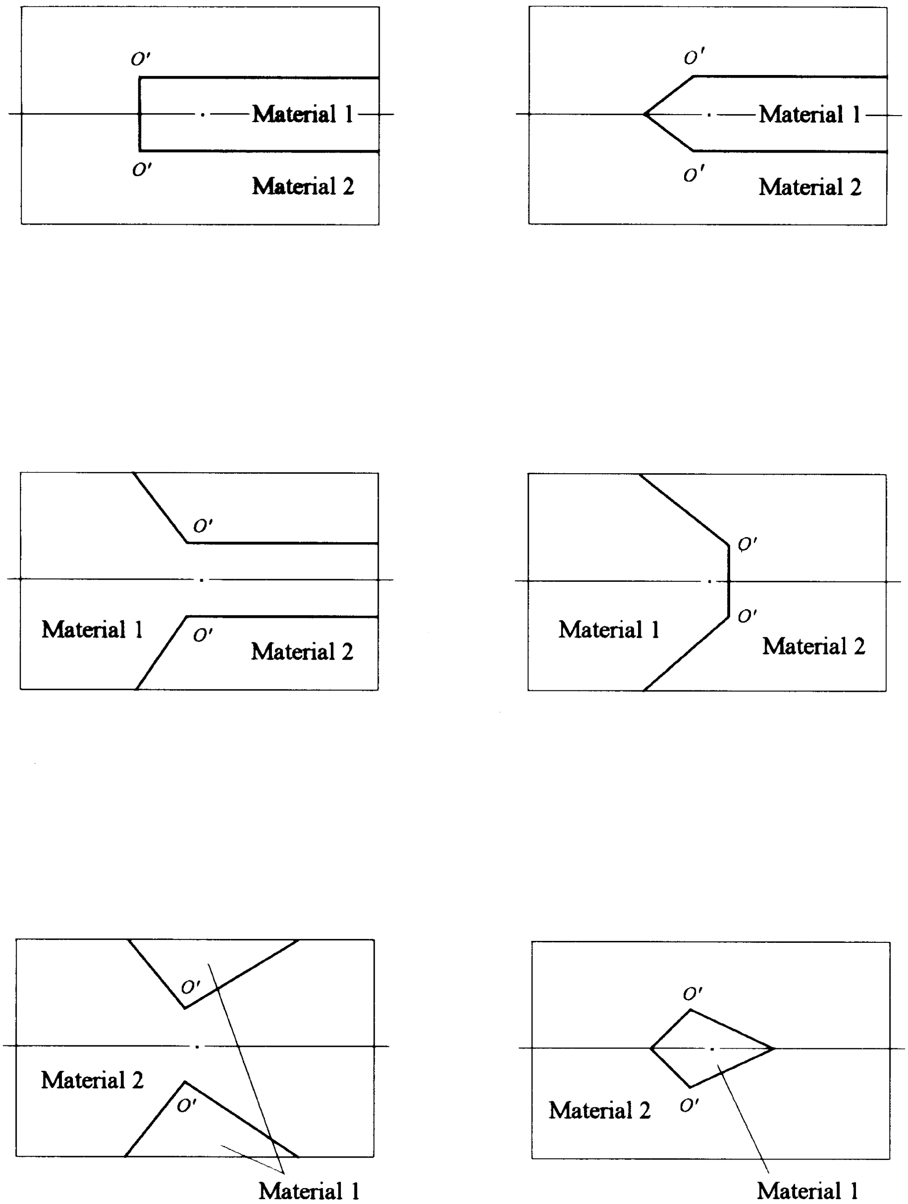

Fig. 2. Some important geometrical shapes.

if the angles θ_0 and θ_1 (Fig. 1) as well as the distance a are selected appropriately, e.g. as is shown in the special cases of Fig. 2. It will be found that the order of the stress singularity in this problem coincides with the result of the corresponding plane strain problem, which (as was already mentioned) was obtained by Bogy and Wang (1971) as well as by Chen and Nisitani (1993). However, the dependence of the singular stress distribution on the material properties is described by the two Poisson ratios (of the two isotropic materials) and the ratio of their shear moduli (i.e. three parameters are needed). Few numerical results (obtained by the finite element method, concerning the axisymmetric three-dimensional elasticity, for a special material combination and in a particular geometry and loading) are also presented and they are compared with the theoretical results obtained in this paper. The theoretical results are also checked by the equilibrium partial differential equations and the boundary conditions on the interfaces. These confirm that the complicated mathematical analysis in this paper is carried out correctly.

2. Basic equations of the axisymmetric elasticity problem

On the basis of the fundamental equations of the axisymmetric three-dimensional isotropic elasticity [in cylindrical coordinates (ρ, ψ, z)], the displacement and stress components can be expressed in terms of two harmonic functions $\phi_j(\rho, z)$ $(j = 1, 2)$, which, evidently, should satisfy the related harmonic partial differential equation (Boussinesq, 1885; Wang, 1988) i.e.

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right)\phi_j = 0 \quad (j = 1, 2). \tag{1}$$

Next, the displacement and stress fields can be represented as

$$2\mu u_\rho = \frac{\partial}{\partial \rho}(\phi_1 + z\phi_2),$$

$$2\mu u_z = \frac{\partial}{\partial z}(\phi_1 + z\phi_2) - 4(1-v)\phi_2. \tag{2}$$

$$\sigma_\rho = \frac{\partial^2}{\partial \rho^2}(\phi_1 + z\phi_2) - 2v\frac{\partial \phi_2}{\partial z},$$

$$\sigma_\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\phi_1 + z\phi_2) - 2v\frac{\partial \phi_2}{\partial z},$$

$$\sigma_z = \frac{\partial^2}{\partial z^2}(\phi_1 + z\phi_2) - 2(2-v)\frac{\partial \phi_2}{\partial z},$$

$$\tau_{\rho z} = \frac{\partial}{\partial \rho}\left[\frac{\partial}{\partial z}(\phi_1 + z\phi_2) - 2(1-v)\phi_2\right] \tag{3}$$

where μ is the shear modulus and v is the Poisson ratio.

In order to examine the displacement and singular stress fields near interface corner O' (Fig. 1), we can use the following coordinate transformation

$$\rho = a + r\sin\theta, \quad \psi = \psi, \quad z = r\cos\theta. \tag{4}$$

Therefore, the harmonic partial differential eqn (1) can be transformed into

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{a+r\sin\theta}\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\right]\phi_j=0 \quad (j=1,2). \tag{5}$$

Furthermore, eqns (2) and (3) can be written as

$$2\mu u_r=\frac{\partial}{\partial r}(\phi_1+\phi_2 r\cos\theta)-4(1-v)\phi_2\cos\theta,$$

$$2\mu u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}(\phi_1+\phi_2 r\cos\theta)+4(1-v)\phi_2\sin\theta. \tag{6}$$

$$\sigma_r=\frac{\partial^2}{\partial r^2}(\phi_1+\phi_2 r\cos\theta)-2\left[(2-v)\cos\theta\frac{\partial}{\partial r}-\frac{v\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\phi_2,$$

$$\sigma_\psi=\frac{1}{a+r\sin\theta}\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(\phi_1+\phi_2 r\cos\theta)-2v\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\phi_2,$$

$$\sigma_\theta=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\phi_1+\phi_2 r\cos\theta)-2\left[v\cos\theta\frac{\partial}{\partial r}-\frac{(2-v)\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\phi_2,$$

$$\tau_{r\theta}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)(\phi_1+\phi_2 r\cos\theta)+2(1-v)\left(\sin\theta\frac{\partial}{\partial r}-\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\phi_2. \tag{7}$$

It can be observed that the harmonic partial differential eqn (5) contains a term with the factor $a+r\sin\theta$ in the denominator. Near the interface corner, it holds true that $r<a$ so that this term can be replaced by a binomial expansion i.e.

$$\frac{1}{a+r\sin\theta}=\frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{r}{a}\sin\theta\right)^k. \tag{8}$$

The substituting of eqn (8) into eqn (5) yields

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{r}{a}\sin\theta\right)^k\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\right]\phi_j=0 \quad (j=1,2). \tag{9}$$

Now let us assume that the harmonic functions $\phi_j(r,\theta)$ can be written in the following series form by the method of separation of variables, r and θ, in each term of the series

$$\phi_j(r,\theta)=\sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{\lambda+2-j+k}g_{jk}(\theta) \quad (j=1,2) \tag{10}$$

where $\lambda(0<\lambda<1)$ is an eigenvalue to be determined, which can be only determined if the specific configuration and boundary/interface conditions are chosen, and $g_{jk}(\theta)$ are unknown functions of the polar angle θ to be determined. To this end, a substituting of eqn (10) into eqn (9) yields

$$g''_{j0}(\theta)+(\lambda+2-j)^2 g_{j0}(\theta)=0,$$

$$g''_{jk}(\theta)+(\lambda+2-j+k)^2 g_{jk}(\theta)$$
$$= -\sum_{n=0}^{k-1}[g'_{jk-n-1}(\theta)\cos\theta+(\lambda+1-j+k-n)g_{jk-n-1}(\theta)\sin\theta](-\sin\theta)^n$$
$$(j=1,2;k=1,2,3,\ldots) \tag{11}$$

where, obviously, the primes denote the differentiations with respect to θ. Now it is easy to solve the ordinary differential eqns (11) and determine $g_{jk}(\theta)$. For $k=0$ and $k=1$ we get

$$g_{j0}(\theta) = A_{j0}\sin(\lambda+2-j)\theta+B_{j0}\cos(\lambda+2-j)\theta,$$
$$g_{j1}(\theta) = A_{j1}\sin(\lambda+3-j)\theta+B_{j1}\cos(\lambda+3-j)\theta$$
$$+\frac{B_{j0}}{4}\sin(\lambda+1-j)\theta-\frac{A_{j0}}{4}\cos(\lambda+1-j)\theta \quad (j=1,2) \tag{12}$$

where A_{j0}, B_{j0}, A_{j1}, B_{j1} are coefficients to be determined from the interface (mainly) and the loading conditions.

Finally, the substituting of the harmonic functions (10) into eqns (6) and (7) yields

$$2\mu u_r = \sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{\lambda+k}\frac{1}{a}[(\lambda+k+1)g_{1k}(\theta)+(\lambda+k-3+4v)ag_{2k}(\theta)\cos\theta],$$

$$2\mu u_\theta = \sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{\lambda+k}\frac{1}{a}[g'_{1k}(\theta)+ag'_{2k}(\theta)\cos\theta+(3-4v)ag_{2k}(\theta)\sin\theta]. \tag{13}$$

$$\sigma_r = \sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{\lambda+k-1}\frac{1}{a^2}[(\lambda+k+1)(\lambda+k)g_{1k}(\theta)$$
$$+2vag'_{2k}(\theta)\sin\theta+(\lambda+k-3+2v)(\lambda+k)ag_{2k}(\theta)\cos\theta],$$
$$\sigma_\psi = \sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{\lambda+k-1}\frac{2v}{a}[g'_{2k}(\theta)\sin\theta-(\lambda+k)g_{2k}(\theta)\cos\theta]$$
$$+\sum_{k=1}^{\infty}\left(\frac{r}{a}\right)^{\lambda+k-1}\frac{1}{a^2}\sum_{n=0}^{k-1}\{[g'_{1n}(\theta)+ag'_{2n}(\theta)\cos\theta]\cos\theta$$
$$+[(\lambda+n+1)g_{1n}(\theta)+(\lambda+n)ag_{2n}(\theta)\cos\theta]\sin\theta\}(-\sin\theta)^{k-n-1},$$
$$\sigma_\theta = \sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{\lambda+k-1}\frac{1}{a^2}[g''_{1k}(\theta)+(\lambda+k+1)g_{1k}(\theta)+ag''_{2k}(\theta)\cos\theta$$
$$+2(1-v)ag'_{2k}(\theta)\sin\theta+(1-2v)(\lambda+k)ag_{2k}(\theta)\cos\theta],$$

$$\tau_{r\theta} = \sum_{k=0}^{\infty} \left(\frac{r}{a}\right)^{\lambda+k-1} \frac{1}{a^2} [(\lambda+k)g'_{1k}(\theta) + (\lambda+k-2+2v)ag'_{2k}(\theta)\cos\theta + (1-2v)(\lambda+k)ag_{2k}(\theta)\sin\theta]. \tag{14}$$

3. The boundary conditions on the interfaces

We consider an axisymmetric interface corner problem of perfectly bonded dissimilar materials (Fig. 1). The regions $\theta_2-\theta_0 \leqslant \theta \leqslant 2\pi-\theta_0$ and $-\theta_0 \leqslant \theta \leqslant \theta_2-\theta_0$ are occupied by the material 1 and 2, respectively. By adopting the above coordinate transformation (4), the displacement and traction continuity conditions on the interface 1 can be written as

$$u_{r1}(r, 2\pi-\theta_0) = u_{r2}(r, -\theta_0), \quad u_{\theta 1}(r, 2\pi-\theta_0) = u_{\theta 2}(r, -\theta_0),$$
$$\sigma_{\theta 1}(r, 2\pi-\theta_0) = \sigma_{\theta 2}(r, -\theta_0), \quad \tau_{r\theta 1}(r, 2\pi-\theta_0) = \tau_{r\theta 2}(r, -\theta_0). \tag{15}$$

On the interface 2 the same continuity conditions take the completely analogous forms

$$u_{r1}(r, \theta_2-\theta_0) = u_{r2}(r, \theta_2-\theta_0), \quad u_{\theta 1}(r, \theta_2-\theta_0) = u_{\theta 2}(r, \theta_2-\theta_0),$$
$$\sigma_{\theta 1}(r, \theta_2-\theta_0) = \sigma_{\theta 2}(r, \theta_2-\theta_0), \quad \tau_{r\theta 1}(r, \theta_2-\theta_0) = \tau_{r\theta 2}(r, \theta_2-\theta_0). \tag{16}$$

In the above sets of equations the subscripts 1 and 2 refer to the material 1 and 2, respectively (Fig. 1).

4. Stress singularity at the interface corner

For convenience, we introduce the subscript m ($m = 1, 2$) to denote the material 1 and 2 (exactly as in the previous section) and introduce the following dimensionless quantity

$$R = \frac{r}{a}. \tag{17}$$

Near the interface corner O' (i.e. as $r \to 0$), we can consider only the first (dominant) terms in eqns (13) and (14) so that the displacement and singular stress fields inside the material m can be represented as

$$u_{rm} = \frac{1}{\mu_m} R^{\lambda} \tilde{h}_{rm}(\theta), \quad u_{\theta m} = \frac{1}{\mu_m} R^{\lambda} \tilde{h}_{\theta m}(\theta) \quad (m = 1, 2) \tag{18}$$

$$\sigma_{rm} = \frac{1}{R^{1-\lambda}} \tilde{f}_{rm}(\theta), \quad \sigma_{\psi m} = \frac{1}{R^{1-\lambda}} \tilde{f}_{\psi m}(\theta),$$

$$\sigma_{\theta m} = \frac{1}{R^{1-\lambda}} \tilde{f}_{\theta m}(\theta), \quad \tau_{r\theta m} = \frac{1}{R^{1-\lambda}} \tilde{f}_{r\theta m}(\theta) \quad (m = 1, 2) \tag{19}$$

where $1-\lambda$ is the order of the stress singularity,

$$
\begin{aligned}
\tilde{h}_{rm}(\theta) = \frac{a}{2}\{&\tilde{A}_{1m}\sin(\lambda+1)\theta+\tilde{B}_{1m}\cos(\lambda+1)\theta \\
&+\tilde{A}_{2m}(\lambda-\kappa_m)[\sin(\lambda+1)\theta+\sin(\lambda-1)\theta] \\
&+\tilde{B}_{2m}(\lambda-\kappa_m)[\cos(\lambda+1)\theta+\cos(\lambda-1)\theta]\}, \\
\tilde{h}_{\theta m}(\theta) = \frac{a}{2}\{&\tilde{A}_{1m}\cos(\lambda+1)\theta-\tilde{B}_{1m}\sin(\lambda+1)\theta \\
&+\tilde{A}_{2m}[(\lambda-\kappa_m)\cos(\lambda+1)\theta+(\lambda+\kappa_m)\cos(\lambda-1)\theta] \\
&-\tilde{B}_{2m}[(\lambda-\kappa_m)\sin(\lambda+1)\theta+(\lambda+\kappa_m)\sin(\lambda-1)\theta]\}.
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\tilde{f}_{rm}(\theta) &= \lambda\{\tilde{A}_{1m}\sin(\lambda+1)\theta+\tilde{B}_{1m}\cos(\lambda+1)\theta \\
&\quad+\tilde{A}_{2m}[(\lambda-\kappa_m)\sin(\lambda+1)\theta+(\lambda-3)\sin(\lambda-1)\theta] \\
&\quad+\tilde{B}_{2m}[(\lambda-\kappa_m)\cos(\lambda+1)\theta+(\lambda-3)\cos(\lambda-1)\theta]\}, \\
\tilde{f}_{\psi m}(\theta) &= -\lambda(3-\kappa_m)[\tilde{A}_{2m}\sin(\lambda-1)\theta+\tilde{B}_{2m}\cos(\lambda-1)\theta], \\
\tilde{f}_{\theta m}(\theta) &= -\lambda\{\tilde{A}_{1m}\sin(\lambda+1)\theta+\tilde{B}_{1m}\cos(\lambda+1)\theta \\
&\quad+\tilde{A}_{2m}[(\lambda-\kappa_m)\sin(\lambda+1)\theta+(\lambda+1)\sin(\lambda-1)\theta] \\
&\quad+\tilde{B}_{2m}[(\lambda-\kappa_m)\cos(\lambda+1)\theta+(\lambda+1)\cos(\lambda-1)\theta]\}, \\
\tilde{f}_{r\theta m}(\theta) &= \lambda\{\tilde{A}_{1m}\cos(\lambda+1)\theta-\tilde{B}_{1m}\sin(\lambda+1)\theta \\
&\quad+\tilde{A}_{2m}[(\lambda-\kappa_m)\cos(\lambda+1)\theta+(\lambda-1)\cos(\lambda-1)\theta] \\
&\quad-\tilde{B}_{2m}[(\lambda-\kappa_m)\sin(\lambda+1)\theta+(\lambda-1)\sin(\lambda-1)\theta]\}
\end{aligned}
\tag{21}
$$

and

$$\kappa_m = 3-4v_m \quad (m = 1,2) \tag{22}$$

$$\tilde{A}_{jm} = \frac{A_{j0m}[\lambda(2-j)+1]}{ja^{3-j}}, \quad \tilde{B}_{jm} = \frac{B_{j0m}[\lambda(2-j)+1]}{ja^{3-j}} \quad (j,m = 1,2) \tag{23}$$

here $\tilde{A}_{jm}$ and $\tilde{B}_{jm}$ $(j,m = 1,2)$ are coefficients to be determined from the interface (mainly) and the loading conditions.

Substituting eqns (18) and (19) into eqns (15) and (16), one obtains a system of eight homogeneous linear algebraic equations for the eight unknown coefficients $\tilde{A}_{jm}$ and $\tilde{B}_{jm}$ $(j,m = 1,2)$. For a non-trivial solution of this system, the solvability condition is simply that the determinant of the matrix of the coefficients be equal to zero. The condition easily leads (after some algebraic computations) to the following eigenequation in λ

$$
\begin{aligned}
\Delta = {}&[\lambda^2(\alpha-\beta)^2\sin^2\theta_1-(1+\beta)^2\sin^2\lambda\theta_1][\lambda^2(\alpha-\beta)^2\sin^2\theta_2-(1-\beta)^2\sin^2\lambda\theta_2] \\
&+(1-\alpha^2)\sin^2\lambda(\pi-\theta_2)[2\lambda^2(\alpha-\beta)^2\sin^2\theta_2+2(1-\beta^2)\sin\lambda\theta_1\sin\lambda\theta_2
\end{aligned}
$$

$$+(1-\alpha^2)\sin^2\lambda(\pi-\theta_2)]=0 \tag{24}$$

where α and β are Dundurs' (1969) bimaterial (composite) constants, which are given by

$$\alpha=\frac{\Gamma(\kappa_1+1)-(\kappa_2+1)}{\Gamma(\kappa_1+1)+(\kappa_2+1)},\quad \beta=\frac{\Gamma(\kappa_1-1)-(\kappa_2-1)}{\Gamma(\kappa_1+1)+(\kappa_2+1)} \tag{25}$$

here the bimaterial constant Γ is simply equal to the ratio of the shear moduli i.e.

$$\Gamma=\frac{\mu_2}{\mu_1}. \tag{26}$$

It is obvious that the eigenvalue λ depends only on the joining angles θ_1 or θ_2 as well as on the bimaterial constants α and β. On the contrary, the interface angle θ_0 (Fig. 1) and the distance a between the interface corner O' and the axisymmetric axis z have no influence on the value of this eigenvalue. Moreover, it should be pointed out that the eigenequation (24) coincides with that valid in the corresponding plane strain problem, which was originally derived by Bogy and Wang (1971) [but see also the more recent derivation by Chen and Nisitani (1993)]. It can also be noted that in the very special case when $\alpha=\beta\neq 0$, the eigenequation (24) takes the much simpler form

$$\Delta=(1-\alpha^2)^2\sin^4\lambda\pi=0 \tag{27}$$

with no root λ in the open interval (0, 1). However, since

$$\Delta|_{\lambda=1}=\left.\frac{d^n\Delta}{d\lambda^n}\right|_{\lambda=1}=0\quad (n=1,2,3) \tag{28}$$

in the same spatial case, a logarithmic singularity may be present (Bogy and Wang, 1971; Dempsey and Sinclair, 1979, 1981).

5. The singular stress field near the interface corner

After the determination of λ as the (smallest positive) solution of the eigenequation (24), one can proceed to the derivation of a non-trivial solution of the aforementioned homogeneous system of eight linear algebraic equations [with its determinant having led to eqn (24)] with respect to the unknown coefficients $\tilde{A}_{jm}$ and $\tilde{B}_{jm}$ in this system. Seven of the unknown coefficients can be expressed in terms of the eighth. This solution can be written as

$$\tilde{A}_{jm}=A^*_{jm}\tilde{B}_{21},\quad \tilde{B}_{jm}=B^*_{jm}\tilde{B}_{21}\quad (j,m=1,2) \tag{29}$$

where $\tilde{B}_{21}$ is (selected as) the only coefficient to be determined (e.g. from the loading at infinity or, in any case, far from the interface corner point O'). After some algebraic computations we finally get

$$A^*_{11}=-\frac{1}{F_1}\left\{\frac{F_3(1)}{F_3(0)}[F_1F_4(0,1,0)-F_2(0,0)F_5(0,0)-F_2(1,0)F_5(1,0)]\right.$$

$$+F_1F_4(1,1,1)+F_2(1,1)F_5(0,0)-F_2(0,1)F_5(1,0)\bigg\},$$

$$A^*_{12} = -\frac{1}{F_1}\left\{\frac{F_3(1)}{F_3(0)}[F_1F_5(0,1)+F_2(0,0)F_4(0,2,1)-F_2(1,0)F_4(1,2,0)]\right.$$

$$\left.-F_1F_5(1,1)-F_2(1,1)F_4(0,2,1)-F_2(0,1)F_4(1,2,0)\right\},$$

$$A^*_{21} = \frac{F_3(1)}{F_3(0)},$$

$$A^*_{22} = \frac{1}{F_1}\left[\frac{F_2(0,0)F_3(1)}{F_3(0)}-F_2(1,1)\right],$$

$$B^*_{11} = \frac{1}{F_1}\left\{\frac{F_3(1)}{F_3(0)}[F_1F_4(1,1,0)-F_2(0,0)F_5(1,0)+F_2(1,0)F_5(0,0)]\right.$$

$$\left.-F_1F_4(0,1,1)+F_2(1,1)F_5(1,0)+F_2(0,1)F_5(0,0)\right\},$$

$$B^*_{12} = \frac{1}{F_1}\left\{\frac{F_3(1)}{F_3(0)}[F_1F_5(1,1)+F_2(0,0)F_4(1,2,1)+F_2(1,0)F_4(0,2,0)]\right.$$

$$\left.+F_1F_5(0,1)-F_2(1,1)F_4(1,2,1)+F_2(0,1)F_4(0,2,0)\right\},$$

$$B^*_{21} = 1,$$

$$B^*_{22} = -\frac{1}{F_1}\left[\frac{F_2(1,0)F_3(1)}{F_3(0)}+F_2(0,1)\right]. \tag{30}$$

The functions F_l ($l = 1,2,3,4,5$) in the above equations (F_1 being simply a constant) are defined by

$$F_1 = 2(1-\alpha)[1-\cos 2\lambda(\pi-\theta_2)],$$

$$F_2(k,n) = \lambda(\alpha-\beta)\left\{\cos 2\left[\frac{k\pi}{4}-(\lambda-1)(\theta_2-\theta_0)\right]-\cos 2\left[\frac{k\pi}{4}-\lambda\theta_2+(\lambda-1)\theta_0\right]\right.$$
$$\left.+\cos 2\left[\left(\lambda-\frac{k}{4}\right)\pi-(\lambda-1)\theta_0\right]-\cos 2\left[\left(\lambda-\frac{k}{4}\right)\pi-\theta_2-(\lambda-1)\theta_0\right]\right\}$$
$$+(-1)^n(1+\beta)\left\{\cos\frac{k\pi}{2}-\cos 2\left[\left(\lambda-\frac{k}{4}\right)\pi-\lambda\theta_2\right]\right.$$
$$\left.+\cos 2\left(\lambda+\frac{k}{4}\right)\pi-\cos 2\left[\left(2\lambda+\frac{k}{4}\right)\pi-\lambda\theta_2\right]\right\},$$

$$F_3(k) = (1+\alpha)F_1\left\{\cos 2\left[\lambda(\theta_2-\theta_0)-\frac{k\pi}{4}\right]-\cos 2\left[\lambda\theta_0-\left(\lambda-\frac{k}{4}\right)\pi\right]\right\}$$

$$+\sum_{n=0}^{1}(-1)^{(1-k)n}F_2(|k-n|,k)\left\{\lambda(\alpha-\beta)\left[\cos 2\left(\theta_2-\theta_0-\frac{n\pi}{4}\right)-\cos 2\left(\theta_0+\frac{n\pi}{4}\right)\right]\right.$$

$$\left.-(-1)^n(1-\beta)\left[\cos 2\left(\lambda\theta_2-\lambda\theta_0-\frac{n\pi}{4}\right)-\cos 2\left(\lambda\theta_0+\frac{n\pi}{4}\right)\right]\right\},$$

$$F_4(k,m,n) = (1-k)(\lambda-\kappa_m)+\lambda\cos 2\left(\theta_2-\theta_0-\frac{k\pi}{4}\right)$$

$$+(-1)^n\frac{1-(-1)^m\beta}{\alpha-\beta}\cos 2\left[\lambda(\theta_2-\theta_0)-\frac{k\pi}{4}\right],$$

$$F_5(k,n) = \frac{1-(-1)^n\alpha}{\alpha-\beta}\cos 2\left[\lambda(\theta_2-\theta_0)-\frac{k\pi}{4}\right]. \tag{31}$$

Next, substituting eqns (29) into eqns (18) and (19), one can obtain the displacement and singular stress fields near the interface corner O' (for the dominant eigenvalue λ)

$$u_{km} = \frac{\tilde{B}_{21}}{\mu_m}R^{\lambda}h^*_{km}(\theta) \quad (m = 1,2) \tag{32}$$

$$\sigma_{klm} = \frac{\tilde{B}_{21}}{R^{1-\lambda}}f^*_{klm}(\theta) \quad (m = 1,2) \tag{33}$$

where $u_{km} = (u_{rm}, u_{\theta m})$, $\sigma_{klm} = (\sigma_{rm}, \sigma_{\psi m}, \sigma_{\theta m}, \tau_{r\theta m})$, $h^*_{km} = (h^*_{rm}, h^*_{\theta m})$, $f^*_{klm} = (f^*_{rm}, f^*_{\psi m}, f^*_{\theta m}, f^*_{r\theta m})$. The angular functions $h^*_{km}(\theta)$ and $f^*_{klm}(\theta)$ can be gotten by replacing the coefficients $\tilde{A}_{jm}$ and $\tilde{B}_{jm}$ $(j,m = 1,2)$ in eqns (20) and (21) by the coefficients A^*_{jm} and B^*_{jm} $(j,m = 1,2)$.

In order to define the normalized stress intensity coefficient, K, indicating the stress intensity near the interface corner O', we let

$$\sigma_{\theta m}(\theta = \theta_2-\theta_0) = \frac{K}{R^{1-\lambda}} \quad (m = 1,2) \tag{34}$$

so that the displacement and singular stress fields near the interface corner O', eqns (32) and (33), can be rewritten as

$$u_{km} = \frac{K}{\mu_m}R^{\lambda}h_{km}(\theta) \quad (m = 1,2) \tag{35}$$

$$\sigma_{klm} = \frac{K}{R^{1-\lambda}}f_{klm}(\theta) \quad (m = 1,2) \tag{36}$$

where $h_{km} = (h_{rm}, h_{\theta m})$, $f_{klm} = (f_{rm}, f_{\psi m}, f_{\theta m}, f_{r\theta m})$ can be given out as

$$h_{km}(\theta) = \frac{h^*_{km}(\theta)}{H} \quad (m = 1, 2) \tag{37}$$

$$f_{klm}(\theta) = \frac{f^*_{klm}(\theta)}{H} \quad (m = 1, 2) \tag{38}$$

here

$$H = f^*_{0m}(\theta_2 - \theta_0) \quad (m = 1 \text{ or } 2). \tag{39}$$

In eqns (35) and (36) it is obvious that K has to be determined by loading conditions (away from O'). As the forms of eqns (35) and (36) are analogous to those in the plane strain problem, K has analogous behaviours in both the plane strain and axisymmetric geometries (see Reedy, 1990, 1991). The check that the singular stress components in eqns (36) satisfy the equilibrium partial differential equations of the three-dimensional axisymmetric elasticity for an isotropic elastic medium as well as the boundary conditions on the interfaces can be found in the Appendix.

On the other hand, for multi-eigenvalue cases, the displacement and singular stress fields can be described by the following sum

$$u_{km} = \frac{1}{\mu_m} \sum_{n=1}^{N} K^{(n)} R^{\lambda_n} h^{(n)}_{km}(\theta) \quad (m = 1, 2) \tag{40}$$

$$\sigma_{klm} = \sum_{n=1}^{N} \frac{K^{(n)}}{R^{1-\lambda_n}} f^{(n)}_{klm}(\theta) \quad (m = 1, 2) \tag{41}$$

where $K^{(n)}$, $h^{(n)}_{km}(\theta)$ and $f^{(n)}_{klm}(\theta)$ denote the stress intensity coefficients to be determined and the angular functions with respect to the eigenvalue λ_n, respectively [with only the N eigenvalues in the interval (0, 1) having been taken into consideration in eqns (40) and (41)].

As remarked by Williams (1959) and Dempsey and Sinclair (1979), the foregoing analysis is available to the case of the complex eigenvalue λ, satisfying the eigenequation (24). In this case, K and $h_{km}(\theta)$ and $f_{klm}(\theta)$ also become complex i.e.

$$\begin{aligned}
&\lambda = \xi + \mathrm{i}\eta, \\
&K = K_1 - \mathrm{i}K_2, \\
&h_{km}(\theta) = h_{km1}(\theta) + \mathrm{i}h_{km2}(\theta), \\
&f_{klm}(\theta) = f_{klm1}(\theta) + \mathrm{i}f_{klm2}(\theta) \quad (m = 1, 2).
\end{aligned} \tag{42}$$

Furthermore, in order to obtain the real displacement and singular stress fields the following real harmonic functions $\phi_{jm}(r, \theta)$ should be taken

$$\phi_{jm}(r, \theta) = \frac{1}{2}[\phi_{jm}(r, \theta, \lambda) + \bar{\phi}_{jm}(r, \theta, \lambda)] \quad (j, m = 1, 2) \tag{43}$$

where a bar denotes (as usual) the complex conjugate of a complex quantity. Now, substituting the (real) harmonic functions (43) into eqns (6) and (7), we can easily obtain the real displacement and singular stress fields as follows

$$
\begin{aligned}
u_{km} &= \frac{1}{2}[u_{km}(\lambda)+\bar{u}_{km}(\lambda)] \\
&= \frac{1}{\mu_m} R^{\xi}\{K_1[\cos(\eta \ln R)h_{km1}(\theta)-\sin(\eta \ln R)h_{km2}(\theta)] \\
&\quad + K_2[\sin(\eta \ln R)h_{km1}(\theta)+\cos(\eta \ln R)h_{km2}(\theta)]\} \quad (m=1,2)
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\sigma_{klm} &= \frac{1}{2}[\sigma_{klm}(\lambda)+\bar{\sigma}_{klm}(\lambda)] \\
&= \frac{1}{R^{1-\xi}}\{K_1[\cos(\eta \ln R)f_{klm1}(\theta)-\sin(\eta \ln R)f_{klm2}(\theta)] \\
&\quad + K_2[\sin(\eta \ln R)f_{klm1}(\theta)+\cos(\eta \ln R)f_{klm2}(\theta)]\} \quad (m=1,2)
\end{aligned} \tag{45}
$$

where $u_{km}(\lambda)$ and $\sigma_{klm}(\lambda)$ correspond to $\phi_{jm}(r,\theta,\lambda)$, $\bar{u}_{km}(\lambda)$ and $\bar{\sigma}_{klm}(\lambda)$ correspond to $\bar{\phi}_{jm}(r,\theta,\lambda)$. It is obvious that in such a case (of a complex eigenvalue λ), the displacement and singular stress field components present oscillating behaviours near the interface corner O' (as is very well known in the literature under the usual and somewhat unrealistic classical elasticity assumptions).

It can easily be found that the dependence of the singular stress field on the material properties depends only on the Poisson ratios $v_{1,2}$ (single-material parameters) as well as on the ratio of the shear moduli Γ, defined by eqn (26). Furthermore, if the eigenvalue λ is single and real, only one undetermined coefficient, K, is included in eqns (35) and (36), this fact indicating that the displacement and singular stress fields in this (simple) case are just one-parameter fields. However, on the contrary, if the same eigenvalue λ is complex, the undetermined coefficient K is complex too and this means that two parameters (coefficients) are needed for the description of the displacement and singular stress fields. Usually, it is rather difficult to proceed to the analytical determination of the coefficient K, but it can be determined by numerical methods (Munz and Yang, 1993).

6. Numerical results and discussion

For a verification of the correctness of the above analytical formulae, the order of the stress singularity at the corner point O' and the angular functions of the stress field components near the corner point O' of a conical diamond inclusion (Fig. 3) was numerically analyzed by the ALGOR FEAS code (ALGOR Corporation), based on the displacement finite element method concerning the axisymmetric three-dimensional elasticity. In Fig. 3, $L = 60$ mm, $W = 30$ mm, $a = 10$ mm, $b = 20$ mm. The material 1 and 2 are quartz and ceramics (Al_2O_3), respectively. Their material constants are $E_1 = 73.1$ GPa, $v_1 = 0.17$ ($\mu_1 = 31.2393$) and $E_2 = 359$ GPa, $v_2 = 0.20$ ($\mu_2 = 149.5833$). According to eqns (25) and (26) one can easily obtain $\alpha = 0.6649$ and $\beta = 0.2681$. Then from eqn (24) the eigenvalue is calculated as $\lambda = 0.7693$. Because of the symmetry of the problem (Fig. 3) only a quarter of the specimen was analyzed. The finite element mesh (in the meridional plane), containing 1685 elements and 1782 nodes, is shown in Fig. 4. It should be pointed out that singular elements corresponding to the stress singularity determined by the

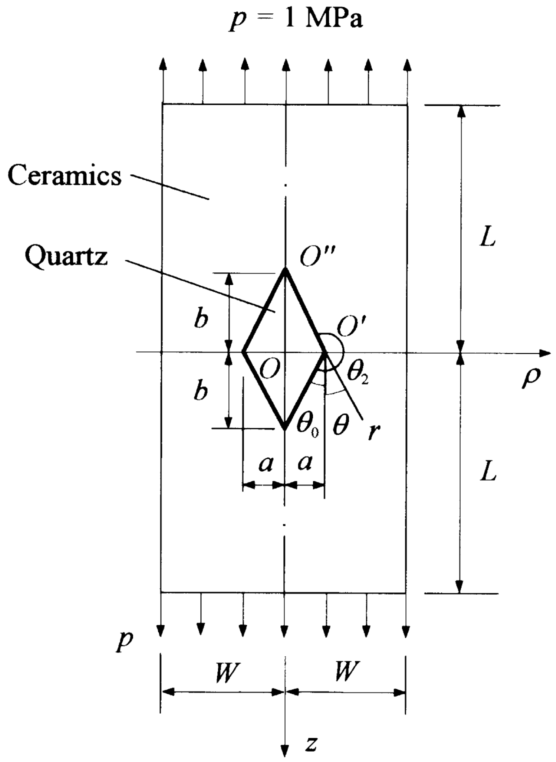

Fig. 3. Tension model of a conical inclusion bonded into a conical notch.

eigenequation (24) should be used for more precise numerical analysis. However, because the order of the stress singularity is examined here, no singular element is used. The minimum finite element length near interface corners O' and O'' is about 10^{-3} mm. The logarithms of the radial distributions of the primary two stress components near the interface corner O' along direction $\theta = 90^\circ$ are shown in Fig. 5. From these distributions the mean value of the order of the stress singularity for the aforementioned stress components is about -0.2223, which is in very good agreement with the theoretical value, -0.2307, obtained from the eigenequation (24), the relative error being less than 3.6%. By the way, it must be pointed out that the values of the stress components on the interface (as these were obtained by the finite element method) are inappropriate for the calculation of the order of the stress singularity because of the stress discontinuity on the interface.

Furthermore, the angular functions $f_\theta(\theta)$ and $f_{r\theta}(\theta)$ normalized by the tangential stress σ_θ at the interface $\theta = \theta_2 - \theta_0$ ($\theta_2 = \pi + 2\theta_0, \theta_0 = \tan^{-1} 0.5$) are also compared at $R = 1.46 \times 10^{-3}$, as shown in Fig. 6. The theoretical values of the angular functions $f_\theta(\theta)$ and $f_{r\theta}(\theta)$ can be obtained from eqn (38) directly. From Fig. 6, it can be found that the angular functions $f_\theta(\theta)$ and $f_{r\theta}(\theta)$ are continuous at the interfaces $\theta = -\tan^{-1} 0.5$ and $\theta = \pi + \tan^{-1} 0.5$, $f_\theta(\theta)$ is symmetrical about $\theta = 90^\circ$ and $\theta = 270^\circ$, however, on the contrary, $f_{r\theta}(\theta)$ is skew-symmetrical about $\theta = 90^\circ$ and $\theta = 270^\circ$. The numerical results coincide essentially with the theoretical results. It should be pointed out that the relative error can be considered as mainly due to the non-accuracy of the interface stress

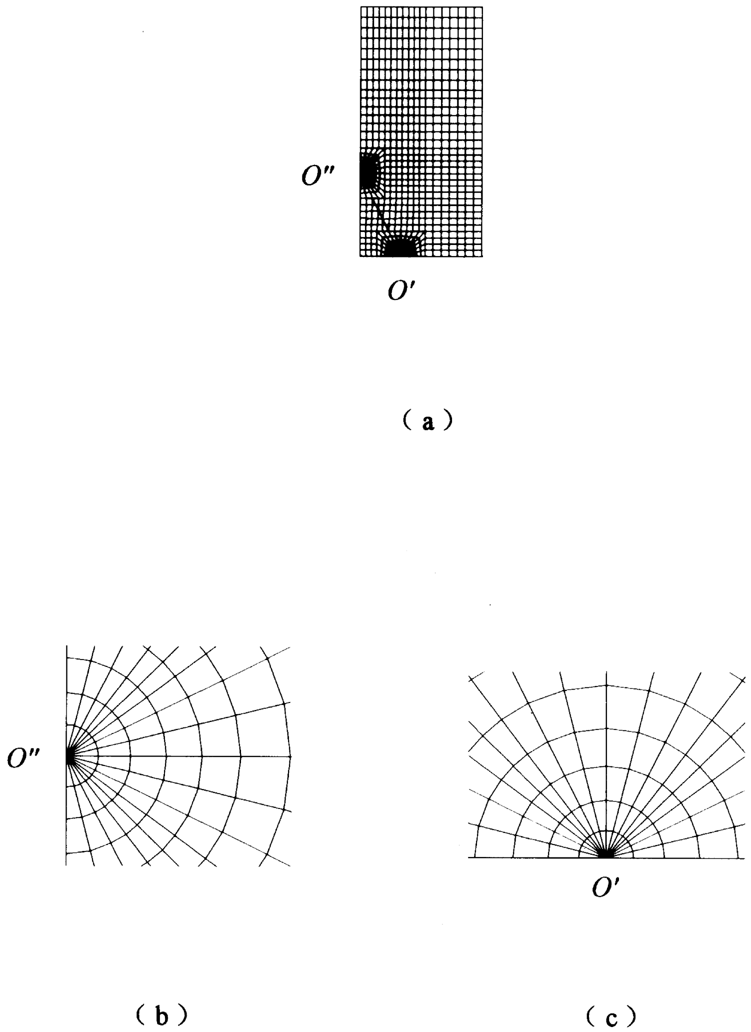

Fig. 4. Calculating model of FEM and its mesh division for (a) Mesh division of finite element method. (b) Local mesh division at the interface corner O''. (c) Local mesh division at the interface corner O'.

$\sigma_\theta(\theta = \pi + \tan^{-1} 0.5)$, obtained by the finite element method, as well as the effect of the high terms. The aforementioned numerical results seem to verify the correctness of the above analytical formulae.

It should be pointed out that the analytical approach, adopted in this paper, is applicable to solve other similar boundary value problems (already having been studied in detail in the plane strain/generalized plane stress singularity problems long ago) in axisymmetric elasticity. Such problems include the interface edge (including the interface crack) (see Naolake and Tomoaki, 1992; Liu et al., 1998), smooth contact [having been solved by Gecit (1974) in terms of the integral transform technique] of two materials. It can also be used to solve the case of three or more materials as well as the case of transversely isotropic materials. However, the solution of the system

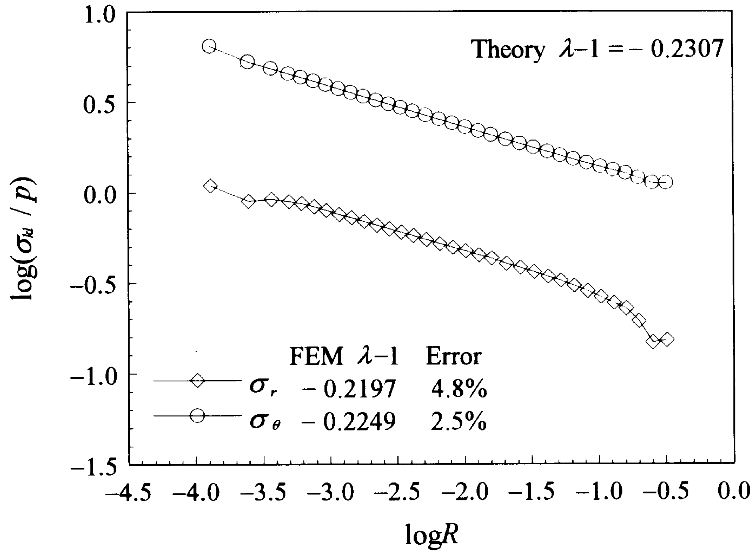

Fig. 5. $\log(\sigma_{kl}/p)$ vs $\log R$ along $\theta = 90°$.

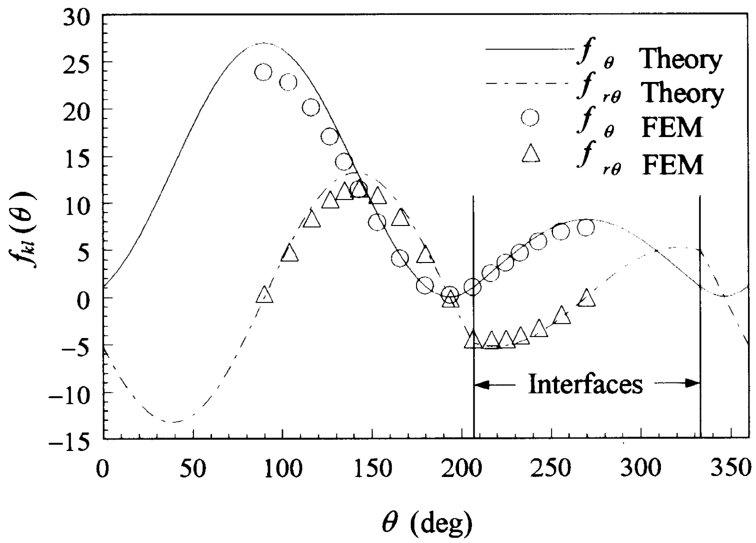

Fig. 6. $f_{kl}(\theta)$ vs θ at $R = 1.46 \times 10^{-3}$.

of homogeneous linear algebraic equations obtaining from the boundary/interface conditions will become difficult as the number of materials increases. In these cases of isotropic materials a slight modification of the boundary/interface conditions (15) and (16) will be needed. In the cases of transversely isotropic materials the related harmonic partial differential equation (1) can be solved by the similar method, adopted in this paper.

On the other hand, as studied by Yuan and Yang (1997) for curved boundary/interface conditions in the plane problems, the singularity results in this paper remain applicable in the case when the two interfaces in Fig. 1 are not straight, but smooth curvilinear, whereas the values of angles θ_0, θ_1 and θ_2 there (in Fig. 1) may remain the same, but now referring to the tangents of the (curvilinear) interfaces at the point O' of their cross-section.

7. Conclusions

The order of the stress singularity and the related displacement and singular stress fields near the axisymmetric interface corner of perfectly bonded dissimilar materials with arbitrary joining angles have been analyzed. The results may have wide applicability to various axisymmetric interface corner problems. It was found that the order of the stress singularity is related only to the joining angles of the interface corner and two Dundurs composite parameters (bimaterial constants) α and β and it coincides with the corresponding result in the plane strain problem, but, on the contrary, the effect of the material properties on the corresponding singular stress field is described by the two Poisson ratios (of the two isotropic materials) and the ratio of their shear moduli. When $\alpha = \beta \neq 0$, instead of the singularity $1/R^{1-\lambda}$, a logarithmic singularity may be present. The numerical results and theoretical check showed that the theoretical results obtained in this paper seem to be correct.

Acknowledgement

The present work was supported by the National Natural Science Foundation of China under Grant No. 19502011.

Appendix: The check of the singular stress field

(1) *The check of the equilibrium equations*

The equilibrium partial differential equations of the three-dimensional axisymmetric elasticity for an isotropic elastic medium [in cylindrical coordinates (ρ, ψ, z)] are well known as follows

$$\frac{\partial \sigma_\rho}{\partial \rho} + \frac{\partial \tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_\psi}{\rho} = 0,$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{\rho z}}{\partial \rho} + \frac{\tau_{\rho z}}{\rho} = 0. \tag{A1}$$

By adopting the coordinate transformation (4), the transformation relations of the stress components between the coordinate systems (ρ, ψ, z) and (r, ψ, θ) can be easily written as

$$\sigma_\rho = \frac{\sigma_r + \sigma_\theta}{2} - \frac{\sigma_r - \sigma_\theta}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta,$$

$$\sigma_\psi = \sigma_\psi,$$

$$\sigma_z = \frac{\sigma_r+\sigma_\theta}{2} + \frac{\sigma_r-\sigma_\theta}{2}\cos 2\theta - \tau_{r\theta}\sin 2\theta,$$

$$\tau_{\rho z} = \frac{\sigma_r-\sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta. \tag{A2}$$

By using eqns (17) and (A2), the equilibrium eqns (A1) can be transformed into

$$\left(\frac{\partial\sigma_r}{\partial R} + \frac{1}{R}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{R}\right)\sin\theta + \left(\frac{1}{R}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial R} + \frac{2\tau_{r\theta}}{R}\right)\cos\theta$$
$$+ \frac{1}{1+R\sin\theta}\left(\frac{\sigma_r+\sigma_\theta}{2} - \frac{\sigma_r-\sigma_\theta}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta - \sigma_\psi\right) = 0,$$

$$\left(\frac{\partial\sigma_r}{\partial R} + \frac{1}{R}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{R}\right)\cos\theta - \left(\frac{1}{R}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial R} + \frac{2\tau_{r\theta}}{R}\right)\sin\theta$$
$$+ \frac{1}{1+R\sin\theta}\left(\frac{\sigma_r-\sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta\right) = 0. \tag{A3}$$

However, if we only consider the first (dominant) terms of the stress field in eqns (14), i.e. only use eqns (36), then only the dominant parts of eqns (A3) are equal to zero, i.e.

$$\frac{\partial\sigma_{rm}}{\partial R} + \frac{1}{R}\frac{\partial\tau_{r\theta m}}{\partial\theta} + \frac{\sigma_{rm}-\sigma_{\theta m}}{R} = 0,$$

$$\frac{1}{R}\frac{\partial\sigma_{\theta m}}{\partial\theta} + \frac{\partial\tau_{r\theta m}}{\partial R} + \frac{2\tau_{r\theta m}}{R} = 0. \tag{A4}$$

It should be noted that this does not mean that the theoretical results are wrong. The remaining parts (high-order terms) in eqns (A3) can be balanced by the high-order terms as shown in eqns (14). In fact, if we use the full form of the stress field as described by eqns (14), the equilibrium eqns (A3) are satisfied perfectly.

(2) *The check of the boundary conditions on the interfaces*

Here we shall only check eqn (16c), i.e. $\sigma_{\theta 1}(r,\theta_2-\theta_0) = \sigma_{\theta 2}(r,\theta_2-\theta_0)$. The check of the others are analogous. Let

$$L = -\frac{R^{1-\lambda}H}{\lambda K}[\sigma_{\theta 1}(r,\theta_2-\theta_0) - \sigma_{\theta 2}(r,\theta_2-\theta_0)] \tag{A5}$$

then the interface continuity condition (16c) means $L = 0$.

First, a substitution of eqns (36) into eqn (A5) yields

$$L = A_1\sin(\lambda+1)(\theta_2-\theta_0) + B_1\cos(\lambda+1)(\theta_2-\theta_0)$$

$$+(\lambda+1)[A_2\sin(\lambda-1)(\theta_2-\theta_0)+B_2\cos(\lambda-1)(\theta_2-\theta_0)] \tag{A6}$$

where

$$A_1 = A_{11}^*+A_{21}^*(\lambda-\kappa_1)-A_{12}^*-A_{22}^*(\lambda-\kappa_2), \quad A_2 = A_{21}^*-A_{22}^*,$$
$$B_1 = B_{11}^*+B_{21}^*(\lambda-\kappa_1)-B_{12}^*-B_{22}^*(\lambda-\kappa_2), \quad B_2 = B_{21}^*-B_{22}^*. \tag{A7}$$

Next, substituting eqns (30) into eqns (A7) we have

$$\begin{aligned} A_1 &= -F\{F_3(1)[F_1a_1-F_2(0,0)a_1+F_2(1,0)b_2] \\ &\quad +F_3(0)[F_1b_2+F_2(0,1)b_2+F_2(1,1)a_1]\}, \\ A_2 &= F\{F_3(1)[F_1-F_2(0,0)]+F_3(0)F_2(1,1)\}, \\ B_1 &= F\{F_3(1)[F_1b_1-F_2(0,0)b_1-F_2(1,0)a_2] \\ &\quad -F_3(0)[F_1a_2+F_2(0,1)a_2-F_2(1,1)b_1]\}, \\ B_2 &= F\{F_3(1)F_2(1,0)+F_3(0)[F_1+F_2(0,1)]\} \end{aligned} \tag{A8}$$

where

$$\begin{aligned} F &= \frac{1}{F_1F_3(0)}, \\ a_k &= \lambda\cos 2(\theta_2-\theta_0)+(-1)^k\cos 2\lambda(\theta_2-\theta_0), \\ b_k &= \lambda\sin 2(\theta_2-\theta_0)+(-1)^k\sin 2\lambda(\theta_2-\theta_0) \quad (k=1,2). \end{aligned} \tag{A9}$$

From eqns (A8) we obtain

$$\begin{aligned} &A_1\sin(\lambda+1)(\theta_2-\theta_0)+B_1\cos(\lambda+1)(\theta_2-\theta_0) \\ &= -(\lambda+1)F\{F_3(1)[F_1-F_2(0,0)]+F_3(0)F_2(1,1)\}\sin(\lambda-1)(\theta_2-\theta_0) \\ &-(\lambda+1)F\{F_3(1)F_2(1,0)+F_3(0)[F_1+F_2(0,1)]\}\cos(\lambda-1)(\theta_2-\theta_0), \\ &A_2\sin(\lambda-1)(\theta_2-\theta_0)+B_2\cos(\lambda-1)(\theta_2-\theta_0) \\ &= F\{F_3(1)[F_1-F_2(0,0)]+F_3(0)F_2(1,1)\}\sin(\lambda-1)(\theta_2-\theta_0) \\ &+F\{F_3(1)F_2(1,0)+F_3(0)[F_1+F_2(0,1)]\}\cos(\lambda-1)(\theta_2-\theta_0). \end{aligned} \tag{A10}$$

Finally, substituting eqns (A10) into eqn (A6) we get

$$L = 0. \tag{A11}$$

Therefore, the interface continuity condition (16c) is satisfied.

References

Atkinson, C., Avila, J., Betz, E., Smelser, R.E., 1982. The rod pull out problem, theory and experiment. Journal of the Mechanics and Physics of Solids 30, 97–120.

Bažant, Z.P., Keer, L.M., 1974. Singularities of elastic stresses and of harmonic functions at conical notches or inclusions. International Journal of Solids and Structures 10, 957–964.

Bogy, D.B., Wang, K.C., 1971. Stress singularities at interface corners in bonded dissimilar isotropic elastic materials. International Journal of Solids and Structures 7, 993–1005.

Boussinesq, M.J., 1885. Applications des Potentiels. Gathier-Villars, Paris.

Chen, D.-H., Nisitani, H., 1992a. Analysis of intensity of singular stress field at fiber end (1st report, method of analysis). Transactions of the Japan Society of Mechanical Engineers (in Japanese) A58, 1834–1838.

Chen, D.-H., Nisitani, H., 1992b. Analysis of intensity of singular stress field at fiber end (2nd report, results of calculation). Transactions of the Japan Society of Mechanical Engineers (in Japanese) A58, 2153–2158.

Chen, D.-H., Nisitani, H., 1993. Singular stress field near the corner of jointed dissimilar materials. ASME Journal of Applied Mechanics 60, 607–613.

Dempsey, J.P., Sinclair, G.B., 1979. On the stress singularities in the plane elasticity of the composite wedge. Journal of Elasticity 9, 373–391.

Dempsey, J.P., Sinclair, G.B., 1981. On the singular behavior at the vertex of a bi-material wedge. Journal of Elasticity 11, 317–327.

Dundurs, J., 1969. Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading [referring to the paper by D. B. Bogy (1968) in the ASME Journal of Applied Mechanics 35, 460–466]. ASME Journal of Applied Mechanics 36, 650–652.

Fleck, N.A., Durban, D., 1989. Asymptotic fields at tip of conical notches and inclusions in a power hardening solid. Journal of the Mechanics and Physics of Solids 37, 233–263.

Gecit, M.R., 1986. Axisymmetric contact problem for a semi-infinite cylinder and a half space. International Journal of Engineering Science 24, 1245–1256.

Gupta, G.D., 1974. The analysis of the semi-infinite cylinder problem. International Journal of Solids and Structures 10, 137–148.

Keer, L.M., Parihar, K.S., 1978. Elastic stress singularity at conical inclusions. International Journal of Solids and Structures 14, 261–263.

Kim, Y.Y., Steele, C.R., 1990. Modifications of series expansions for general end conditions and corner singularities on the semi-infinite strip. ASME Journal of Applied Mechanics 57, 581–588.

Kim, Y.Y., Steele, C.R., 1992. Static axisymmetric end problems in semi-infinite and finite solid cylinders. ASME Journal of Applied Mechanics 59, 69–76.

Kohno, Y., Ishikawa, H., 1991. Singular stress and displacement fields around the corner point of a fiber end in short fiber reinforced composites. International Journal of Engineering Science 29, 1005–1012.

Liu, Y.-H., Xu, J.-Q., Ding, H.-J., 1998. Singular stress field near axisymmetric interface edge of bonded dissimilar materials. ASME Journal of Applied Mechanics, in press.

Muki, R., Sternberg, E., 1970. Elastostatic load-transfer to a half-space from a partially embedded axially loaded rod. International Journal of Solids and Structures 6, 69–90.

Munz, D., Yang, Y.Y., 1993. Stress near the edge of bonded dissimilar materials described by two stress intensity factors. International Journal of Fracture 60, 169–177.

Naolake, N., Tomoaki, T., 1992. Stress singularities in edge-bonded dissimilar wedges (three-dimensional axisymmetric elastic problems). Transactions of the Japan Society of Mechanical Engineers (in Japanese) A58, 275–278.

Pak, R.Y.S., Gobert, A.T., 1993. Axisymmetric problems of a partially embedded rod with radial deformation. International Journal of Solids and Structures 30, 1745–1759.

Picu, C.R., 1996. Stress singularities at vertices of conical inclusions with freely sliding interfaces. International Journal of Solids and Structures 33, 2453–2457.

Reedy, E.D. Jr, 1990. Intensity of the stress singularity at the interface corner between a bonded elastic and rigid layer. Engineering Fracture Mechanics 36, 575–583.

Reedy, E.D. Jr, 1991. Intensity of the stress singularity at the interface corner of a bonded elastic layer subjected to shear. Engineering Fracture Mechanics 38, 273–281.

Robert, M., Keer, L.M., 1987. An elastic circular cylinder with displacement prescribed at the ends-axially symmetric case. Quarterly Journal of Mechanics and Applied Mathematics 40, 339–363.

Somaratna, N., Ting, T.C.T., 1986a. Three-dimensional stress singularities at conical notches and inclusions in transversely isotropic materials. ASME Journal of Applied Mechanics 53, 89–96.

Somaratna, N., Ting, T.C.T., 1986b. Three-dimensional stress singularities in anisotropic materials and composites. International Journal of Engineering Science 24, 1115–1134.

Wang, M.Z., 1988. On the completeness of solutions of Boussinesq, Timpe, Love, and Michell in axisymmetric elasticity. Journal of Elasticity 19, 85–92.

Williams, M.L., 1959. The stresses around a fault or crack in dissimilar media. Bulletin of the Seismological Society of America 49, 199–204.

Yuan, F.G., Yang, S., 1997. The curved interfacial crack between dissimilar isotropic solids. International Journal of Solids and Structures 34, 641–660.

Zak, A.R., 1972. Elastic analysis of cylindrical configurations with stress singularities. ASME Journal of Applied Mechanics 39, 501–506.